

## A Special Form of Galerkin's Method Applied to Heat Transfer in Plane Couette-Poiseuille Flows

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The heat-transfer problem for plane Couette-Poiseuille flows is solved by a Galerkin's procedure for which the system of representing functions has a close relation to the governing equation. A coordinate transformation brings the principal part of the partial differential equation into a standard form, for which the eigenfunctions can be determined once and for all. The resulting system of ordinary differential equations has strong diagonal dominance and is integrated by a predictor corrector method for stiff equations. The accuracy of the method is examined by a comparison of the eigenfunctions of Galerkin's operator with those of the exact problem. The dimensionless temperature field has been computed for various pressure gradients.

### 1. INTRODUCTION

A steady laminar incompressible plane Couette-Poiseuille flow in which heat conduction in the flow direction and viscous dissipation are negligible is governed by the following equation for the dimensionless temperature  $T(x, y)$ :

$$f(\alpha, y)(\partial T/\partial x) = \partial^2 T/\partial y^2, \quad 0 \leq y \leq 1.0. \quad (1)$$

Here  $f(\alpha, y)$  is a dimensionless velocity which is determined from the momentum equation. One finds

$$f(\alpha, y) = 8[y + 2\alpha y(1 - y)]. \quad (2)$$

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The parameter  $\alpha$  is related to the pressure gradient. The definition of  $\alpha$  and  $x$  used here agrees with that of [1]. The present approach is not restricted to the specific form of  $f(\alpha, y)$  defined above. In order for the problem to be well posed  $f(\alpha, y)$  must be nonnegative. [A zero of  $f(\alpha, y)$  at  $y = 0$  and  $y = 1$  is admissible.] The following boundary and initial conditions will be imposed:

$$T(x, 0) = T_0(x), \quad T(x, 1) = T_1(x), \quad (3)$$

$$T(0, y) = T_i(y). \quad (4)$$

To obtain solutions of (1) under general boundary conditions it is necessary to employ a numerical method. In [2] an equation of this type has been used as a model in the study of numerical methods for the solution of the boundary layer equation. The present paper is an outgrowth of these investigations. The underlying idea will be described in Sections 2 and 3. The method can be interpreted as a modification of the classical approach used in [1]. It is advantageous if the problem must be solved for a number of functions  $f(\alpha, y)$ .

## 2. SELECTION OF APPROXIMATING FUNCTIONS

A number of methods for solving the problem (1)–(4) can be considered as special forms of Galerkin's procedure. Let  $\hat{g}_k(y)$ ,  $k = 1, 2, \dots$ , be a complete set of known functions and let

$$T(x, y) = \sum_k \gamma_k(x) \hat{g}_k(y) + f_1(x, y), \quad (5)$$

where  $f_1(x, y)$  is chosen to satisfy the boundary conditions. One is then led to solve the system of equations

$$\frac{d\gamma_j}{dx} + \sum_k c_{jk} \gamma_k = s_j(x), \quad (6)$$

where  $s_j(x)$  are expressed in terms of  $f_1(x, y)$  and the approximating functions  $\hat{g}_k(y)$ . Equation (6) can be put in the vector form

$$d\Upsilon/dx + C\Upsilon = \mathbf{s}, \quad (7)$$

where  $C$  is a constant matrix with elements  $c_{jk}$ .

The classical method of separation of variables can be considered as a variant of Galerkin's procedure for which the system of functions  $\hat{g}$  possesses a close

relation to the partial-differential Eq. (1). The functions  $g$  are obtained by the product hypothesis

$$T(x, y) = \exp(-\lambda x) g(y). \quad (8)$$

One is led to the eigenvalue problem

$$d^2g/dy^2 + \lambda f(\alpha, y)g = 0, \quad g(0) = g(1) = 0, \quad (9)$$

which has to be solved repeatedly for different values of  $\alpha$  [1]. In analogy to (5) we write

$$T(x, y) = \sum_k \beta_k(x) g_k(y) + f_1(x, y). \quad (10)$$

This leads to a system of uncoupled equations

$$d\beta_j/dx + \lambda_j \beta_j = r_j(x). \quad (11)$$

Let  $A$  be a diagonal matrix whose elements are given by  $\lambda_j$ . Then the vector form of (11) is

$$d\beta/dx + A\beta = \mathbf{r}. \quad (12)$$

If in (7) a product hypothesis analogous to (8), that is,

$$\gamma(x) = \exp(-\hat{\lambda}_k x) \mathbf{z}_k, \quad (13)$$

is made, then one obtains an eigenvalue problem for the infinite matrix  $C$ . Denote by  $\hat{\lambda}_k$  the eigenvalues and by  $\mathbf{z}_k$  the eigenvectors; let  $z_{k,m}$  be the  $m$ -th component of  $\mathbf{z}_k$ . The two hypotheses, (5) and (10), then give the same solution and

$$g_k(y) = \sum_j z_{k,j} g_j(y). \quad (14)$$

Of course, this is true only if one considers infinite matrices. If truncated matrices are considered, then their eigenvalues will differ somewhat from the exact ones; very little for small  $\lambda_k$ , more for larger  $\lambda_k$ . Naturally if  $C$  is truncated to an  $N$  by  $N$  matrix, then only  $N$  eigenvalues will be found. It can be expected that the  $k$ -th eigenvalue of the truncated and of the complete matrix have about the same magnitude. This has an important consequence for the numerical approach. All  $\lambda_k$  are positive, therefore the system is stable. But one must differentiate between the stability of the system and the stability of the integration procedure. The stability of the integration procedures usually used (e.g., Runge-Kutta or one of the customary predictor-corrector methods) depends upon the largest eigenvalue of the matrix  $C$ . If large eigenvalues are present then the stability of the integration procedure requires the use of a small step size. This is true even if the solution itself has practically no components which correspond to the large eigenvalues.

Systems with large eigenvalues are called stiff. Every integration procedure for the problem (1)–(4) must cope with its stiffness in some fashion.

Several ideas have been advanced for integrating stiff systems with a step size which is not too small. One can, for instance, bring the system into its diagonal form. In the present context this approach is practically the same as separation of variables. Here the individual equations can be solved by quadratures. The effect of large eigenvalues is present but it appears in manageable form; it manifests itself in the rapid changes of some of the integrands. If the integration is carried out with usual techniques, e.g., the trapezoidal rule or Simpson's rule, then small steps must again be used in order to retain accuracy; large steps are admissible if one anticipates the changes of the integrands by an analytic technique. The drawback of this approach lies in the necessity to determine the eigenvectors of the matrix  $C$ , or (preferably, in the present case), the eigenfunctions  $g_k$ . Other approaches for which larger integration steps are admissible are based on implicit integration methods. Implicit methods require the solution of linear systems of equations for each integration step, but they are quite effective.

In [3] an integration procedure for stiff equations has been developed which is advantageous, if all large elements of  $C$ , that is, those elements by which the stiffness expresses itself, occur in the main diagonal. For such matrices the stiff components are only weakly coupled with the rest of the system and the integration can be carried out by a predictor-corrector method with automatic step control. A form of  $C$  which has this property is obtained, if the asymptotic representations of the eigenfunctions  $g_k$  are used as the approximating functions  $\hat{g}_k$  of Galerkin's method.

### 3. METHOD OF SOLUTION

We carry out a transformation which is suggested by the derivation of Langer's asymptotic representations for the eigenfunctions  $g_k$  [4]. This transformation is applied to the original partial-differential Eq. (1), rather than to the equation for the eigenfunctions. We want the principal part of the governing equation to assume a standard form; this involves a transformation of the  $y$  coordinate and a trivial change of the scale of the  $x$  coordinate. We set

$$T(x, y) = h(y) \phi(\xi, \eta(y)), \quad \xi = Kx. \quad (15)$$

The functions  $h(y)$  and  $\eta(y)$  must be determined in a suitable manner. Substituting (15) into (1) and denoting derivatives with respect to  $y$  by primes we find after division by  $h\eta'^2$

$$K \frac{f(\alpha, y)}{(\eta')^2} \frac{\partial \phi}{\partial \xi} = \frac{\partial^2 \phi}{\partial \eta^2} + \frac{1}{\eta'} \left( 2 \frac{h'}{h} + \frac{\eta''}{\eta'} \right) \frac{\partial \phi}{\partial \eta} + \frac{1}{(\eta')^2} \frac{h''}{h} \phi. \quad (16)$$

A normalized form of the equation is obtained by postulating that  $Kf/(\eta')^2$  be some standard function of  $\eta$ . We restrict ourselves to the case where the only zero of  $f(\alpha, y)$  occurs at  $y = 0$  and choose for this standard form

$$K[f(\alpha, y)/(\eta')^2] = \eta \quad (17)$$

with the additional requirement that  $\eta = 1$  for  $y = 1$ . We then find

$$\eta = K^{1/3} \left[ \frac{3}{2} \int_0^y f^{1/2} dy \right]^{2/3}, \quad K = \left[ \frac{3}{2} \int_0^1 f^{1/2} dy \right]^{-2}. \quad (18)$$

Setting the coefficient of the derivative  $\phi_\eta$  to zero, one finds

$$h(y) = (\eta')^{-1/2}. \quad (19)$$

The coefficient of  $\phi$  in (16) is then given by

$$q(\eta) = \frac{1}{(\eta')^2} \frac{h''}{h} = (\eta')^{-4} \left[ \frac{3}{4} (\eta'')^2 - \frac{1}{2} \eta' \eta''' \right]. \quad (20)$$

For  $y = 0$ ,  $h(y)$  and  $q(\eta)$  are regular functions, but this is not immediately obvious from (19) and (20). Formulas for computing  $h$  and  $q$  in which differences of large numbers are not encountered, even for very small values of  $y$ , are derived in the Appendix.

We now have to deal with the differential equation

$$\eta(\partial\phi/\partial\xi) = (\partial^2\phi/\partial\eta^2) + q(\eta)\phi, \quad 0 \leq \eta \leq 1.0 \quad (21)$$

with the associated boundary and initial conditions

$$\phi(\xi, 0) = \phi_0(\xi) = T_0(x(\xi))/h(0), \quad \phi(\xi, 1) = \phi_1(\xi) = T_1(x(\xi))/h(1), \quad (22)$$

$$\phi(0, \eta) = \phi_i(\eta) = T_i(y(\eta))/h(y(\eta)). \quad (23)$$

For Galerkin's procedure we use approximating functions  $e_k$  defined by the equation

$$d^2e/d\eta^2 + \mu\eta e = 0, \quad e(0) = e(1) = 0. \quad (24)$$

They are asymptotic representations for the functions  $g_k$  defined by (9). For the functions  $e_k$  one has orthonormality conditions of the form

$$\int_0^1 \eta e_j(\eta) e_k(\eta) d\eta = \delta_{jk}. \quad (25)$$

The advantage of this procedure lies in the fact, that one can determine the approxi-

mate eigenfunctions  $e_k$  once and for all; they are independent of the functions  $f$ . Of course, the computation of the functions  $\eta(y)$ ,  $h(y)$ , and  $q(y)$  requires some additional work, but in comparison with the determination of a sufficient number of eigenfunctions and eigenvalues for each  $f$  this is a minor task.

Following Galerkin's procedure we now write

$$\phi(\xi, \eta) = \sum_{k=1}^N \gamma_k(\xi) e_k(\eta) + (1 - \eta) \phi_0(\xi) + \eta \phi_1(\xi). \quad (26)$$

*Remark.* Simpler equations might have been obtained by the hypothesis

$$\phi(\xi, \eta) = \sum_{k=1}^N \gamma_k(\xi) e_k(\eta) + \frac{1}{h(\eta)} [(1 - y) \phi_0(\xi) + y \phi_1(\xi)]$$

for  $T = y$  and  $T = 1 - y$  satisfies the original differential equation. If  $\phi_0$  and  $\phi_1$  are constant, then the inhomogeneous terms would vanish entirely.

Substituting (26) into (21) and using (24) and (25) one is led to the system of equations written in the vector form

$$d\gamma/d\xi + M\gamma = A\gamma + \mathbf{b}(\xi), \quad (27)$$

where  $M$  is a diagonal matrix with elements  $\mu_j$  defined by (24); the elements of the matrix  $A$  and the components of the vector  $\mathbf{b}$  are

$$A_{jk} = \int_0^1 q(\eta) e_j(\eta) e_k(\eta) d\eta, \quad (28)$$

$$\begin{aligned} b_j(\xi) = & \phi_0(\xi) \int_0^1 (1 - \eta) q e_j d\eta - \frac{d\phi_0}{d\xi} \int_0^1 \eta(1 - \eta) e_j d\eta + \phi_1(\xi) \int_0^1 \eta q e_j d\eta \\ & - \frac{d\phi_1}{d\xi} \int_0^1 \eta^2 e_j d\eta. \end{aligned} \quad (29)$$

The initial values for the components of the vector  $\gamma$  are given by

$$\gamma_j(0) = \int_0^1 \phi_i(\eta) \eta e_j(\eta) d\eta - \phi_0(0) \int_0^1 \eta(1 - \eta) e_j d\eta - \phi_1(0) \int_0^1 \eta^2 e_j d\eta. \quad (30)$$

The coefficients  $A_{jk}$  arise from the function  $q(\eta)$ . This function is bounded and the eigenfunctions  $e_j$  and  $e_k$  are bounded (because of the conditions of orthogonality). Therefore the elements  $A_{jk}$  are bounded. In contrast some of the values  $\mu_k$  will be large, unless one limits the number of eigenfunctions rather severely. Thus, (27) constitutes a system for which the method of [3] is advantageous. Incidentally, if the matrix  $M$  is not dominant (perhaps if the elements  $A_{jk}$  are large),

the method can still be carried out, the accuracy control will then lead to small integration steps. In no case will the method of [3] have a detrimental effect.

Our transformation leads in a natural manner to the asymptotic representations of the eigenfunctions. The prerequisite for the application of [3] is diagonal dominance in the governing matrix. This goal can also be achieved if one choose  $h(y) = 1$ . One is then led to the equation

$$\eta(\partial\phi/\partial\xi) = \partial^2\phi/\partial\eta^2 + P(\eta)(\partial\phi/\partial\eta), \quad P(\eta) = \eta''/(\eta')^2. \quad (31)$$

For higher eigenfunction the term  $P(\partial\phi/\partial\eta)$  will be larger than the term  $q\phi$  of (21) (because of the differentiation of  $\phi$ ), but this fact need not be detrimental to the integration process. By the hypothesis

$$\phi(\xi, \eta) = \sum_{k=1}^N \hat{\gamma}_k(\xi) e_k(\eta) + (1 - \eta) T_0(\xi) + \eta T_1(\xi) \quad (32)$$

we are now led to the equations in vector form

$$d\hat{\gamma}/d\xi + M\hat{\gamma} = \hat{A}\hat{\gamma} + \hat{\mathbf{b}}(\xi) \quad (33)$$

in which the elements of  $\hat{A}$  and  $\hat{\mathbf{b}}$  are

$$\begin{aligned} \hat{A}_{jk} &= \int_0^1 P(\eta) e_j(\eta) \frac{de_k}{d\eta} d\eta, \\ b_j(\xi) &= -T_0(\xi) \int_0^1 P e_j d\eta - \frac{dT_0}{d\xi} \int_0^1 \eta(1 - \eta) e_j d\eta + T_1(\xi) \int_0^1 P e_j d\eta \\ &\quad - \frac{dT_1}{d\xi} \int_0^1 \eta^2 e_j d\eta. \end{aligned} \quad (34)$$

The initial values for (33) are given by

$$\hat{\gamma}_j(0) = \int_0^1 \eta T_i(\eta) e_j(\eta) d\eta - T_0(0) \int_0^1 \eta(1 - \eta) e_j d\eta - T_1(0) \int_0^1 \eta^2 e_j d\eta. \quad (36)$$

#### 4. SOME REMARKS ABOUT THE INTEGRATION METHOD

The special integration method to which we have referred above is analyzed in detail in [3]. The important feature is the representation of the right-hand side of (27) by a polynomial of a given degree (the program has been written for degree 4). The effect of the matrix  $M$  is taken into account analytically. It is then

possible to carry out the integration by a predictor-corrector scheme. If the matrix  $M$  on the left is incorporated into the matrix  $A$  on the right or if  $M \equiv 0$ , then one obtains the customary predictor-corrector schemes; if  $A \equiv 0$  then one obtains the analytic solutions and the method is stable for any integration step. Therefore one expects to have stability for a fairly large integration step if  $A$  is small in comparison to  $M$ . A check on the accuracy, i.e., on the validity of the assumption that the right-hand sides can be approximated by polynomials, is made by comparing the predicted and the corrected values. This check is used for interval control. The numerical results are based on the accuracy of  $10^{-4}$  for the truncation error in the prediction-correction phase.

## 5. DISCUSSION OF THE METHOD

Any numerical solution will contain certain errors. If we use a development in terms of eigenfunctions, they are caused by the inaccuracies which arise in the representation of the initial conditions and possibly of inhomogeneous terms by a finite number of terms of their eigenfunction expansion. These are the only errors, besides rounding errors; the operator which gives the homogeneous part of the differential equation is left unchanged.

In Galerkin's method this operator will be modified, for the procedure is limited to a finite number of terms. The relation between the exact and the approximate operators can be analyzed by a comparison of their eigenfunctions and eigenvalues. We preface this comparison by some general remarks. The deviations of the asymptotic representations from the actual eigenfunctions depend upon the closeness of the function  $f(\alpha, y)$  to the standard function (in our case  $\eta$ ) which replaces it [see (21)]. If these functions are too different, then  $q$  will be large and  $h$  will vary strongly. But if  $q$  is large, then  $\mu$  must be very large, otherwise  $q$  cannot be neglected in comparison to  $\mu\eta$ . This observation is of course borne out by the error estimate which is made in the rigorous treatment of asymptotic representations. If  $f(\alpha, y)$  has a zero of the first order then the standard function which takes its place must have a zero at the corresponding point, otherwise  $h$  and  $q$  will be infinite at this point. The standard function  $\eta$  of (21) takes into account the zero of  $f(\alpha, y)$  which occurs for  $y = 0$ , but it does not allow for a second zero which would be encountered in a Poiseuille flow. If  $f(\alpha, y)$  comes close to a second zero, as happens for large values of  $\alpha$ , then  $q$  assumes extremely large values, and the asymptotic approximations become unreliable. The functions on  $\eta$ ,  $h$ ,  $q$ , and  $P$  for different values of  $\alpha$  are shown in Figs. 1-4.

The product hypothesis (13) leads to the eigenvalue problem

$$(C - \hat{\lambda}_k I) \mathbf{z}_k = 0, \quad (37)$$



where  $C = M - A$  for (27),  $I$  is the unit matrix and  $\mathbf{z}_k$  is a vector with the components  $z_{k,1}, z_{k,2}, \dots, z_{k,N}$ . The eigenfunctions pertaining to this operator are

$$\tilde{g}_k(y) = h(y) \sum_{j=1}^N z_{k,j} e_j(\eta(y)). \quad (38)$$

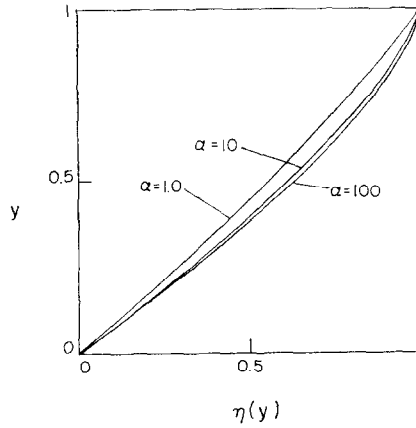


FIG. 1. The function  $\eta(y)$  for different  $\alpha$ .

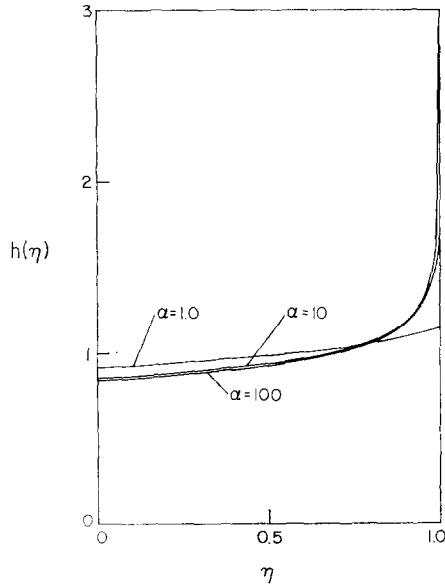


FIG. 2. The function  $h(\eta)$  for different  $\alpha$ .

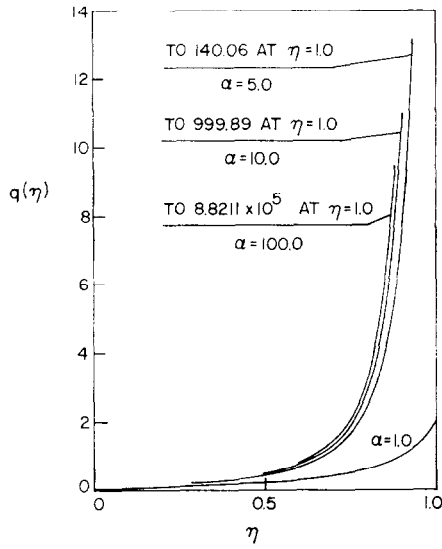


FIG. 3. The function  $q(\eta)$  for different  $\alpha$ .

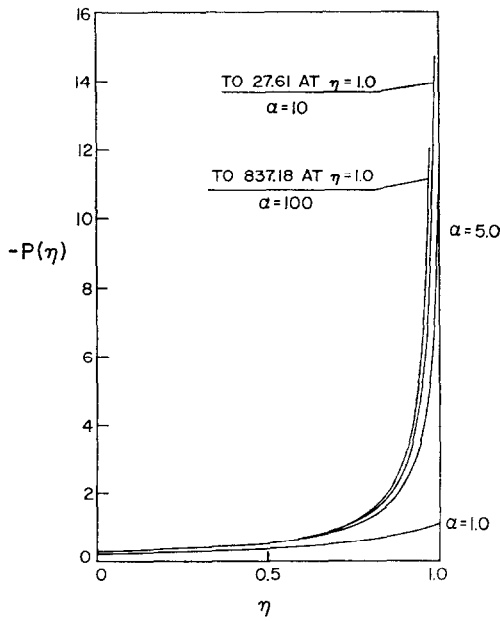


FIG. 4. The function  $P(\eta)$  for different  $\alpha$ .

The accuracy of the procedure is determined by the accuracy with which the eigenfunctions  $\tilde{g}_k$  and the eigenvalues  $\tilde{\lambda}_k$  approximate the corresponding quantities of the exact operator. The same comparison can be carried out for the case where  $h = 1$ , i.e.,  $C = M - \hat{A}$ .

In our computations we have chosen  $N = 10$ . For a number of values  $\alpha$ , the eigenvalues and eigenvectors for the matrices in (27) and in (33) have been determined. Table I lists the exact eigenvalues and different approximations for  $\alpha = 1$  and 10. Further computations for  $\alpha = 25$  and  $\alpha = 50$  have shown that the difference between the eigenvalues obtained with  $h(y) = (\eta')^{-1/2}$  and with  $h = 1$  is never more than 0.3 %.

TABLE I  
The Eigenvalues Obtained from the Different Methods

Mode	$\alpha = 1.0$				$\alpha = 10.$		
	Airy's Eq. (24)	$h = (\eta')^{-1/2}$	$h(y) = 1.0$	Exact	$h = (\eta')^{-1/2}$	$h(y) = 1.0$	Exact
1	18.95	18.38	18.38	18.38	16.53	16.52	16.51
2	81.88	81.21	81.21	81.21	77.47	77.45	77.43
3	189.2	188.5	188.5	188.5	183.3	183.2	183.2
4	341.0	340.2	340.2	340.3	333.8	333.7	333.7
5	537.1	536.4	536.4	536.4	528.9	528.8	528.8
6	777.7	777.0	777.0		768.6	768.3	
7	1063.	1062.	1062.		1053.	1052.	
8	1392.	1391.	1391.		1382.	1381.	
9	1766.	1765.	1765.		1755.	1753.	
10	2185.	2183.	2182.	2183.	2173.	2168.	2172.

For  $\alpha = 10$  the difference between the 5-th and the 10-th eigenfunctions and various approximations is shown in Figs. 5 and 6. The best results are obtained from (37) and (38) with  $h = (\eta')^{-1/2}$ . For the lower modes the approximations with  $h = 1$  are nearly as good, and both are better than the asymptotic approximation. For higher modes the approximation with  $h = 1$  deteriorates. The accuracy of the approximation decreases with increasing  $\alpha$ . Figure 7 gives the same information for the derivatives. In view of the pronounced peak that occurs in  $q(y)$  the approximations are surprisingly good. The function  $f(y)$  is shown in Fig. 8; it deviates very considerably from the linear function which takes its place in (21). The requirement that the operators of the approximating equations should be "close" to that of the exact equation is certainly not very stringent. From these discussions we conclude that  $\alpha = 10$  is a safe, perhaps even conservative, limit for the application of the present procedure.

The procedure of [3] is stable for fairly large integration steps if, in those equa-

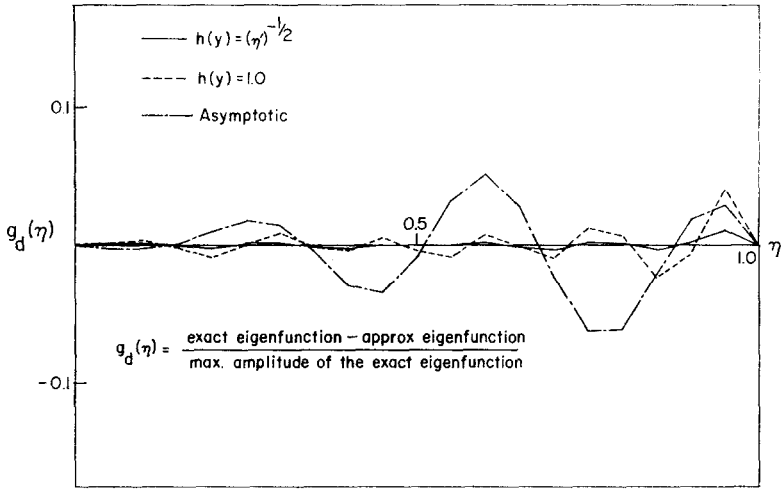


FIG. 5. Different approximations to the fifth eigenfunction for  $\alpha = 10$ .

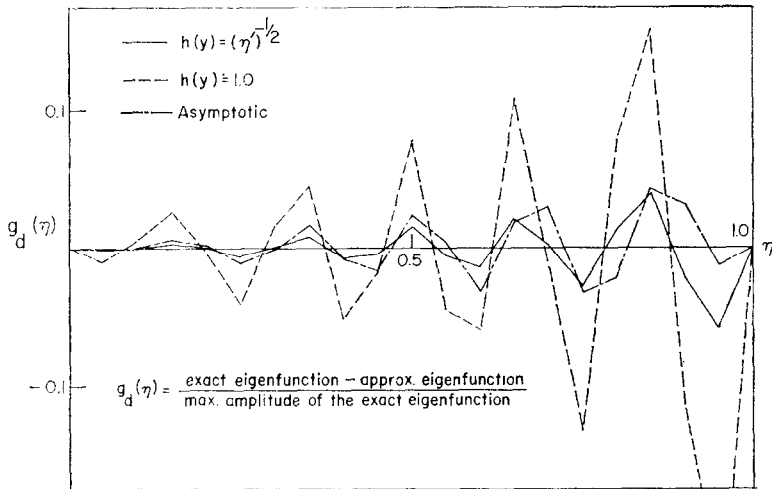


FIG. 6. Different approximations to the tenth eigenfunction for  $\alpha = 10$ .

tions for which  $\mu_j$  is large, the matrix elements  $A_{jk}$  or  $\hat{A}_{jk}$  are small in comparison to the  $\mu_j$ . For the systems (27) and (33) these matrix elements are indeed small. Even for  $\alpha = 100$  the worst value of  $\hat{A}_{jk}$  is less than 17% of the corresponding  $\mu_j$  and this happens for a low value of  $\mu_j$ .

At the start of the integration procedure a step size  $10^{-3}$  was chosen. This is

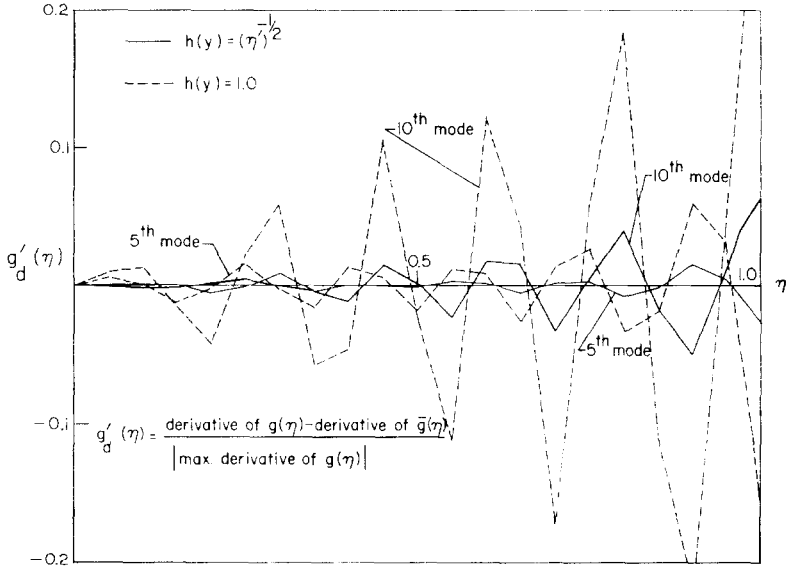


FIG. 7. Different approximations to the derivative of the fifth and tenth eigenfunction for  $\alpha = 10$ .

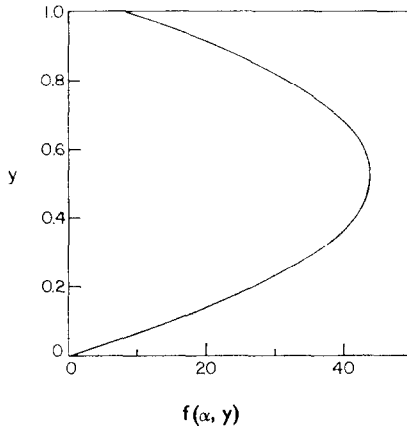


FIG. 8. Velocity profile,  $f(\alpha, y)$ , for  $\alpha = 10$ .

larger than the stability limit for the ordinary predictor-corrector method. For a largest eigenvalue of about 2200, this limit would be  $4.4 \times 10^{-4}$ . But for the procedure of [3] the stability limits are much higher, therefore the provisions of the program for automatic step control have increased the step size a number of times. At  $\xi = 1$  we obtained a step of  $3.2 \times 10^{-2}$  or sometimes  $6.4 \times 10^{-2}$ .

If the computation would have been carried out further, larger step sizes would probably have been used. The step size seems to be limited by accuracy requirements rather than stability requirements. (Actually the program has no stability check. But instabilities lead to a deterioration of the accuracy, and then the step size is reduced because of the accuracy requirements. If the step size is close to the stability limit then a program with automatic step control will periodically change the step size from a stable value to an unstable value and back.)

## 6. RESULTS

As a first application we have considered the problem solved by Hudson and Bankoff [1] by means of an eigenfunction development. The boundary conditions are

$$T_0(x) = T_1(x) = 0, \quad T_i(y) = 1.0. \quad (39)$$

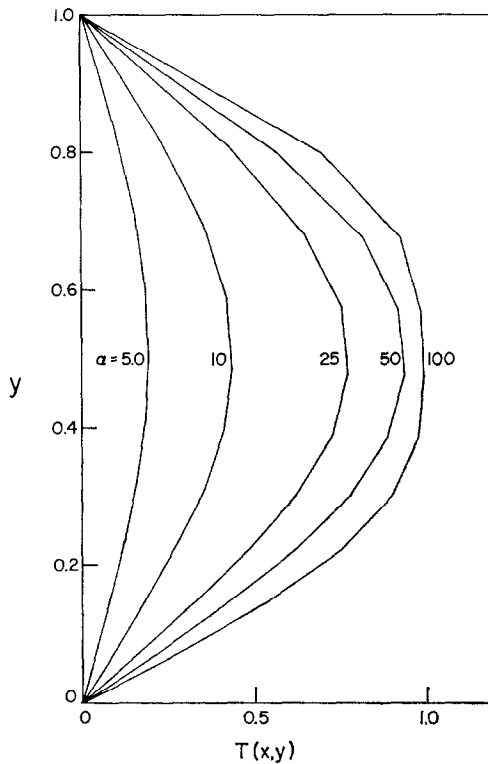


FIG. 9. Temperature profiles at  $x = 4.0$ .

In these examples no inhomogeneous term appears in the differential equation. Since the present method gives excellent representations for the eigenfunctions and eigenvalues it is not surprising that there is excellent agreement between our results and the curves of [1]. In Fig. 9 temperature profiles at  $x = 4$  for a number of values of  $\alpha$  are shown. One recognizes the influence of  $\alpha$  on the thermal entrance length. For  $\alpha < 1$  the temperatures at  $x = 4$  are practically zero.

In a second set of examples we assumed that the temperature at the upper plate differs from that of the lower plate, i.e., we impose as boundary conditions

$$T_0(x) = 0, \quad T_1(x) = 1.0, \quad T_i(y) = 1.0. \quad (40)$$

As we mentioned above, the expression  $T = y$  satisfies the inhomogeneous boundary conditions and the exact partial-differential equation. The rest of the problem is therefore identical with the previous example except for the initial temperature distribution. However, our computations have been carried out with the hypothesis (26) and then inhomogeneous terms appear. They are different for the case  $h = 1$  and the case  $h \neq \text{const}$ . For this reason differences in the solution especially for large values of  $\alpha$  are encountered. However, for small values of  $\xi$  results of the two methods agree rather well; they are within 2% accuracy even for  $\alpha = 100$  at  $x = 2.0$ . For Couette flow,  $\alpha = 0$ , the results given in Fig. 10 agree with those presented graphically in [5], as they should for in this case the methods are the same. Temperature profiles at different stations  $x$  for  $\alpha = 1, 5$ , and 10 are shown in Figs. 11–13. Figure 14 shows temperature profiles for different values of  $\alpha$ . One recognizes again the influence of  $\alpha$  on the thermal entrance length.

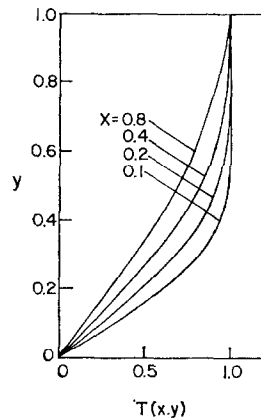
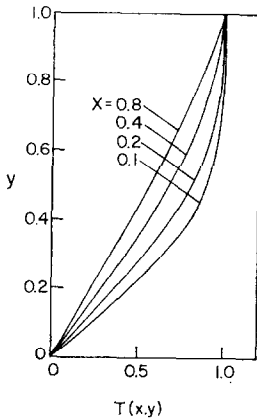


FIG. 10. Temperature profiles for  $\alpha = 0$ . FIG. 11. Temperature profiles for  $\alpha = 1.0$ .

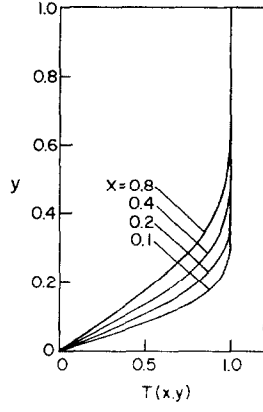
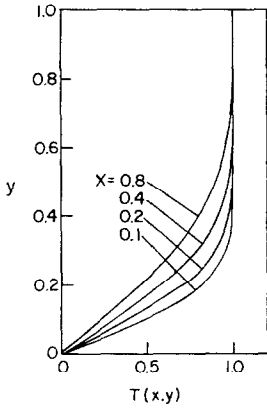


FIG. 12. Temperature profiles for  $\alpha = 5$ . FIG. 13. Temperature profiles for  $\alpha = 10$ .

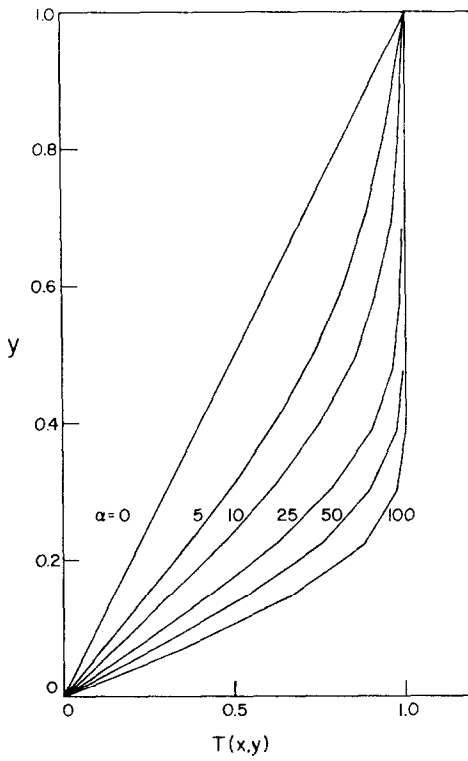


FIG. 14. Temperature profiles at  $x = 2.0$ .



For large values of  $\xi$  where the initial condition is not important the temperature distribution should be given by  $T = y$ . We found that the solutions with  $h = 1.0$  are very satisfactory even when  $\alpha = 100$ , while the solutions with  $h = (\eta')^{-1/2}$  show discrepancies for large values of  $\alpha$ . These errors are introduced by the approximation of the inhomogeneous terms; the inhomogeneous term considered as a function of  $y$  is much more peaked if we use the method with  $h = (\eta')^{-1/2}$  than if  $h = 1.0$ . The shortcoming which expresses itself by these inaccuracies should not be ascribed to the method, but to the inadequacy of the hypothesis (26). Inadvertently this hypothesis uses a function for the representation of the boundary conditions for which a development with respect to  $g_k$  or  $\hat{g}_k$  converges very poorly. If other representations are used, for instance, the function  $T = y$  mentioned above, then one would obtain very satisfactory results.

Finally we would like to point out that the present analysis can as well be applied to more complicated problems such as the heat transfer problem with viscous dissipation [that is the case when Eq. (1) involves a term of known function of  $y$ ] and incompressible boundary-layer flow problems.

#### APPENDIX: FORMULAS FOR $h(y)$ AND $q(y)$

Differentiating (18) one obtains

$$\eta' = \eta f^{1/2} / \left[ \frac{3}{2} \int_0^y f^{1/2} dy \right]$$

or

$$\eta' = (2/3)(\eta u_1' / u_1) \quad (\text{A.1})$$

with

$$u_1 = \int_0^y f^{1/2} dy, \quad u_1' = f^{1/2}. \quad (\text{A.2})$$

Assume that  $f(y)$  has the form

$$f(y) = a_1 y + a_2 y^2 + a_3 y^3 + \dots, \quad a_1 \neq 0. \quad (\text{A.3})$$

Certain expressions related to  $f$  are difficult to compute for small values of  $y$ , because of singularities which occur in the analytic expressions. We want to derive formulas where this difficulty does not occur. One has

$$u_1 = O(y^{3/2}).$$

The integration involved in the definition of  $u_1$  can be carried out numerically

without a loss of relative accuracy. It follows that no difficulties are encountered in the evaluation of  $\eta'$  and  $h$ .

The expression (20) for  $q(y)$  can be written as

$$q(y) = (\eta')^{-2} \left[ \frac{1}{4} \left( \frac{\eta''}{\eta'} \right)^2 - \frac{1}{2} \frac{d}{dy} \left( \frac{\eta''}{\eta'} \right) \right]. \quad (\text{A.4})$$

We derive expressions for the evaluation of the individual terms in the bracket. From (A.1) by logarithmic differentiation one finds

$$\frac{\eta''}{\eta'} = \frac{\eta'}{\eta} + \frac{u_1''}{u_1'} - \frac{u_1'}{u_1}.$$

Hence using the definition of  $u_1$ , (A.2), the above expression becomes

$$\frac{\eta''}{\eta'} = \frac{1}{2} \frac{f'}{f} - \frac{1}{3} \frac{u_1'}{u_1}. \quad (\text{A.5})$$

The individual terms on the right side of this equation are  $O(y^{-1})$  (for small  $y$ ) while the entire expression is  $O(1)$ . We try to split off the singular parts by a systematic procedure. To do this for the first term on the right consider

$$yf' - f = yf' - \int_0^y f' dy.$$

Next by integration by part

$$yf' - f = u_2$$

with

$$u_2 = \int_0^y yf'' dy = O(y^2). \quad (\text{A.6})$$

Then it follows

$$\frac{f'}{f} - \frac{1}{y} = \frac{u_2}{yf} = O(1). \quad (\text{A.7})$$

In this form the singular part of  $f'/f$  is displayed analytically. The essence of the procedure is seen more clearly if one deals with the second term in (A.5). The power series development suggests that

$$\frac{u_1'}{u_1} - \frac{3}{2} \frac{1}{y} = O(1).$$

We consider accordingly

$$\begin{aligned} yu_1' - \frac{3}{2}u_1 &= yf^{1/2} - \frac{3}{2}\int_0^y f^{1/2} dy \\ &= yf^{1/2} - \frac{3}{2}\int_0^y \left(\frac{f}{y}\right)^{1/2} y^{1/2} dy = \int_0^y \frac{d}{dy} \left(\frac{f^{1/2}}{y^{1/2}}\right) y^{3/2} dy. \end{aligned}$$

By logarithmic differentiation one has

$$yu_1' - \frac{3}{2}u_1 = \frac{1}{2}\int_0^y yf^{1/2} \left(\frac{f'}{f} - \frac{1}{y}\right) dy.$$

Next, by use of (A.7) one obtains

$$yu_1' - (3/2)u_1 = (1/2)u_3$$

with

$$u_2 = \int_0^y u_2 d^{-1/2} dy = O(y^{5/2}). \quad (\text{A.8})$$

Hence

$$\frac{u_1'}{u_1} - \frac{3}{2}\frac{1}{y} = \frac{1}{2}\frac{u_3}{yu_1} = O(1). \quad (\text{A.9})$$

Substituting (A.7) and (A.9) into (A.5) one finds

$$\frac{\eta''}{\eta'} = \frac{1}{2}\frac{u_2}{yf} - \frac{1}{6}\frac{u_3}{yu_1}. \quad (\text{A.10})$$

Starting from (A.10) one finds by the same technique

$$\frac{d}{dy} \left(\frac{\eta''}{\eta'}\right) = \frac{1}{2} \left( \frac{u_4}{y^2f} - \frac{u_2^2}{y^2f^2} \right) - \frac{1}{6} \left( \frac{u_5}{y^2u_1} - \frac{1}{2} \frac{u_3}{y^2u_1^2} \right) \quad (\text{A.11})$$

$O(1) \quad O(1) \quad O(1) \quad O(1)$

with

$$u_4 = \int_0^y y^2f' dy, \quad u_5 = \int_0^y (u_4 f^{-1/2} - \frac{1}{2}u_2^2 f^{-3/2}) dy. \quad (\text{A.12})$$

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